# Recognition of Plane Projective Symmetry

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## Abstract

A novel approach to grouping symmetrical planar curves under a projective transform is described. Symmetric curves are important as a generic model for object recognition where an object class is defined by the set of symmetries that any object in the class obeys. In this paper, a new algorithm is presented for grouping curves based on their correspondence under a plane projectivity. The correspondence between curves is established from an initial correspondence between two pairs of distinguished lines, such as lines tangent to inflection points. This initial correspondence leads to a reduced dimensional form for the projective mapping between the curves and a natural method for establishing correspondence between all points on the curves. A saliency measure is introduced which permits grouping results to be ordered in terms of the degree of symmetry supported by each curve pair. This saliency measure provides a basis for recognition in the case of approximate symmetry.

## 1 Introduction

### 1.1 The Process of Recognition

In order to recognize an object in a cluttered scene with highly textured and complex backgrounds, and with partial occlusion of the object by other objects in the scene, it is necessary to carry out two functions: figure-ground separation and classification. Often these two functions are inter-twined as in model-based vision where the projection of a specific 3-d object is used to group image features corresponding to the model.

Such specific grouping mechanisms can be effective but do not readily lead to a recognition process that can handle general classes of objects. Techniques which rely on specific, three dimensional, geometric models of the objects to be recognized are difficult to adapt to the rapid pace of change in the real world and cannot easily support the in-class variability of real object structure.

Generic models are vital to the future of object recognition. It has long been recognized that symmetry is a well-defined, intuitively accessible generic model[2]. Symmetry is pervasive in imagery because a symmetrical object is both statically and dynamically more stable. For this reason, even natural objects such as flowers and trees exhibit a high degree of symmetry.

### 1.2 What is Symmetry?

Symmetry can be considered in the most general way as a relationship between two or more geometric structures according to the following definition.

> Transformational symmetry is defined by a transformation group, $G$ and a set of geometric structures, $\mathcal{S}$, where for each pair of structures, $s_i, s_j \in \mathcal{S}$, $\exists\, g \in G$, such that $s_j = g(s_i)$ and $s_i = g^{-1}(s_j)$.

In this paper, we restrict the class of symmetries to plane projective transformations. While not all symmetries of interest can be described in this restricted framework, it is still quite general and will establish the required grouping and recognition infrastructure.

### 1.3 Related Work

The work here is most closely related to a number of efforts directed at the use of generic model-based grouping constraints to extract object descriptions from scenes. There has been extensive study of grouping outlines of rotationally symmetric objects[13] and

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of generalized cylinders[12]. In these investigations, the constraints imposed by the 3-d surface geometry of the class induces a planar symmetry constraint in the image. In practice, the grouping was carried out using an affine approximation to the projective symmetry constraints.

The general symmetry constraint defined by an affine symmetry transformation was exploited by Cham and Cipolla[5]. They did not relate the constraint to a specific object class but demonstrated that affine symmetries can be grouped successfully in complex scenes. They also introduced the notion of *saliency* which measures the significance of a match between a pair of curves. The formulation of a saliency metric is necessary in order to avoid trivial symmetries.

Leung and Malik have investigated the detection of structures repeated under planar affine transformations[9]. They demonstrate the matching of texture descriptions to recover symmetric features.

Fleck et al. have used generic axial symmetry descriptions and the geometric relations between symmetry axes to distinguish the parts of the human body[7]. The axes are recovered under the assumption that smoothed local symmetry holds. This work emphasizes the strong link between object class and the grouping constraints induced by the class.

## 2 Plane Projective Symmetries

In this paper, plane projective, rather than affine, symmetry is implemented with projectively invariant descriptions. More precisely, the general definition of symmetry given in Section 1.2 is restricted as follows.

> Plane projective symmetry is defined by the set of plane projective transformation matrices $\mathcal{G}$ and a set of planar geometric structures, $\mathcal{S}$, where for each pair of structures, $s_i, s_j \in \mathcal{S}$, $\exists$ T $\in G$, such that $s_j = \pi(s_i, \mathrm{T})$ and $s_i = \pi(s_j, \mathrm{T}^{-1})$. Here $\pi$ is a plane projective transformation of the structure.

Beyond actual planar symmetries such as wallpaper and fabric patterns viewed under perspective, a wide variety of three dimensional objects also include the plane projective symmetry constraint in images, for example, objects of continuous rotational symmetry.

In this paper, the development is focused on planar curve features. A large class of objects is captured by the observation that the planar transformation constraint between curve segments does not require that both curves are co-planar. Figure 1 shows the relation between two curves in 3-d, $\mathbf{C}$ and $\mathbf{C}'$. The image projections of $C$ and $C'$ are related by a projectivity.

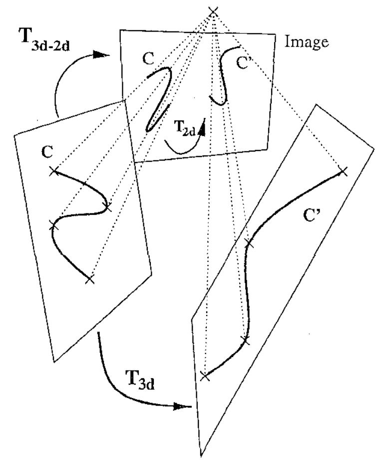


Figure 1: Here planar curves $\mathbf{C}$ and $\mathbf{C}'$ are related by a plane projection $\mathrm{T}_{3d}$, and both are viewed as images $C$ and $C'$ under perspective projection, $\mathrm{T}_{3d-2d}$. The transform from $C$ to $C'$ in the image is also a projectivity, $\mathrm{T}_2d$.

Many man-made and natural structures have planar curve sections, such as the petals of a flower or the surface intersections of manufactured objects. However, while these features are not necessarily co-planar they do usually have some symmetry relation, such as translation, rotation or scale. The projection of these 3-d symmetry transforms into the image still produces a plane projective transformation and therefore serves as a source of grouping constraint.

### 2.1 Classification of Projective Symmetries

Plane projective symmetries can be classified in terms of the eigenvalues of the projection matrix T. The eigenvectors of the matrix define points and lines in the image which are fixed under the transformation according to various conditions that might hold among the eigenvalues. Such a classification has been made by Springer[10]. VanGool has constructed a useful interpretation of these symmetry classes in terms of the geometry of the fixed structures[11]. The rank of the transformation matrix must be three, since each symmetry transformation must have an inverse. Several allowable general constraints among the eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, and their interpretations are given in Table 1. As an example of interpreting the fixed struc-

| Constraint | Symmetry Class | Fixed Structures |
|---|---|---|
| $\lambda_1$ real; $\lambda_2$ real; $\lambda_3$ real; | general projectivity | three fixed points |
| $\lambda_1$ real; $\lambda_2$, $\lambda_3$ complex conjugates; | rotation | one fixed point and a line of two fixed points |
| $\lambda_2 = \lambda_3$; | planar homology | a pencil of fixed lines and a fixed line |
| $\lambda_2 = \lambda_3; \lambda_1 = -\lambda_2$; | harmonic homology | same as above with $T^2 = I$ |

Table 1: A classification of plane projective transformations.

tures, the transformation matrix for the harmonic homology can be represented by,

$$T = I - 2\,\frac{\mathbf{p}\,\mathbf{l}^T}{\mathbf{l}^T\,\mathbf{p}} \tag{1}$$

Here $\mathbf{p}$ is the center of the fixed pencil, and $\mathbf{l}$ is the line of fixed points. A similar construction has been devised by Viéville[1]. A harmonic homology has period two, i.e., $T^2 = I$.

### 2.2 Affine Symmetries

Much of the prior work has concentrated on grouping under affine symmetry, defined as follows.

> Plane affine symmetry is defined by the set of plane affine transformation matrices $\mathcal{G}$ of the form,
>
> $$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ 0 & 0 & 1 \end{bmatrix}$$
>
> and a set of planar geometric structures, $\mathcal{S}$, where for each pair of structures, $s_i, s_j \in \mathcal{S}$, $\exists\ T \in G$ such that $s_j = \alpha(s_i, T)$ and $s_i = \alpha(s_j, T^{-1})$. Here $\alpha$ is a plane affine transformation of the structure.

This more restricted symmetry is widely exploited for grouping, since it is a good approximation to perspective image projection when objects are viewed from a distance large compared to their depth. For the case of rotationally symmetric objects it is necessary to have a very wide angle lens before perspective effects become significant, even for very close-up views.

[1] A. Zisserman, private communication

## 3 Curve Grouping under Projective Symmetry

### 3.1 The Grouping Process

The symmetry grouping process involves three distinct procedural steps:

1. Decide which pair of curves, $C$ and $C'$ are to be matched.
2. Determine the correspondence between any point on $C$ and its symmetrical image on $C'$, and vice versa.
3. Evaluate the match to determine the degree of symmetry.

Steps 1 and 3 are somewhat inter-related and discussion of them is deferred until Section 4. In this section the problem of determining a point by point mapping of one curve onto another is considered.

### 3.2 Point Correspondence

If the symmetry transformation between the curves is Euclidean, then the point correspondence can be established, up to an unknown translation along the curve, simply by parameterizing each curve by arc length. In the case of projective transformations, more parameters are needed to establish point correspondence.

In order to reduce the combinatorial complexity of determining these parameters, the invariance of special curve features is exploited. For example in grouping rotational symmetric object outlines, the bitangent feature is used to robustly extract distinguishable points on the contour[13]. Here, tangent lines at inflection points are used to initiate the point correspondence mapping. An inflection point is locally collinear and thus tangency at inflection points is preserved under projectivities.

### 3.3 Tangent Lines At Two Inflection Points

Consider two inflection points $\mathbf{i}_1$ and $\mathbf{i}_2$ on $C$. Construct the tangent lines $\mathbf{l}_1$ and $\mathbf{l}_2$ at these points. Assuming these lines are distinct, they define a basis for a pencil of lines $P$ passing through their intersection.

Any line $\mathbf{l}$ in $P$ may be written as a linear combination of $\mathbf{l}_1$ and $\mathbf{l}_2$:

$$\mathbf{l} = \alpha \mathbf{l}_1 + \beta \mathbf{l}_2.$$

Now consider any point $\mathbf{p}$ on the curve $C$, and construct the line in $P$ passing through $\mathbf{p}$. This line must satisfy $\mathbf{l}^T\mathbf{p} = 0$ and $\mathbf{l} = \alpha\mathbf{l}_1 + \beta\mathbf{l}_2$. Together, these imply

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\mathbf{l}_2^T\mathbf{p} \\ \mathbf{l}_1^T\mathbf{p} \end{bmatrix} \tag{2}$$

The parameters $\alpha$ and $\beta$ may be viewed in two ways. First, as illustrated in Figure 2a, they may be viewed as the homogeneous coordinates of the line through $\mathbf{p}$ in the one-dimensional projective space formed by the pencil of lines. Second, they may be viewed as the coordinates of $\mathbf{p}$ following an affine transformation:

$$\begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathbf{l}_2^T \\ \mathbf{l}_1^T \\ 0 \ 0 \ 1 \end{bmatrix} \mathbf{p} = \mathtt{L}\mathbf{p}. \tag{3}$$

Both of these views are necessary to establish correspondence between points on corresponding curves.

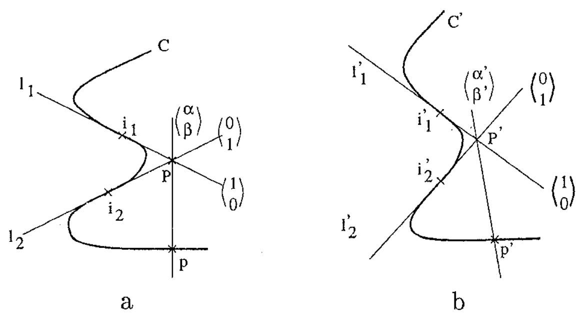


Figure 2: (a) A pencil of lines defined by the tangents at two inflection points, $\mathbf{l}_1$ and $\mathbf{l}_2$, to the curve $C$. The homogeneous coordinates of a line within this pencil, $[\alpha, \beta]^T$, are defined in the basis formed by the two tangent lines. Any point $\mathbf{p}$ on the curve $C$ is assigned a coordinate $[\alpha, \beta]^T$ based on the line in the pencil passing through $\mathbf{p}$. Referring to both (a) and (b), if the line defined by $[\alpha, \beta]^T$ corresponds to the line defined by $[\alpha', \beta']^T$, then the intersection points of these lines with the curves must also correspond.

### 3.4 Correspondence in Pencil Space

Let $C$ and $C'$ be two curves related by a projectivity $\mathtt{T}$. For an arbitrary parameterization of $C$, point $\mathbf{p}(s)$ on $C$ corresponds to point $\mathbf{p}'(s)$ on $C'$ and, using homogeneous coordinates,

$$\mathbf{p}'(s) = \lambda_p \mathtt{T}\mathbf{p}(s). \tag{4}$$

Here $\lambda_p$ represents the scaling ambiguity of homogeneous coordinates. The inflection point tangents on $C$ will also be transformed projectively:

$$\begin{aligned} \mathbf{l}'_1 &= \lambda_1 \mathtt{T}^{-T}\mathbf{l}_1 \\ \mathbf{l}'_2 &= \lambda_2 \mathtt{T}^{-T}\mathbf{l}_2 \end{aligned} \tag{5}$$

where $\mathtt{T}^{-T}$ is the transpose of the inverse of $\mathtt{T}$. Here $\mathbf{l}'_i$ is the line corresponding to $\mathbf{l}_i$. Thus if the correspondence of the tangents at inflection points is assumed, we can set up a pencil space for each curve $P(\mathbf{l}_1, \mathbf{l}_2)$ and $P(\mathbf{l}'_1, \mathbf{l}'_2)$. Now $\mathbf{p}(s)$ and $\mathbf{p}'(s)$ have coordinates $[\alpha(s), \beta(s)]^T$ and $[\alpha'(s), \beta'(s)]^T$ given by equation 2:

$$\begin{aligned} \begin{bmatrix} \alpha'(s) \\ \beta'(s) \end{bmatrix} &= \begin{bmatrix} -\mathbf{l}'^T_2\mathbf{p}'(s) \\ \mathbf{l}'^T_1\mathbf{p}'(s) \end{bmatrix} \\ \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} &= \begin{bmatrix} -\mathbf{l}_2^T\mathbf{p}(s) \\ \mathbf{l}_1^T\mathbf{p}(s) \end{bmatrix} \end{aligned}$$

We can solve these equations to give the relation between the curves in the pencil spaces.

$$\begin{bmatrix} \alpha'(s) \\ \beta'(s) \end{bmatrix} = \begin{bmatrix} \lambda_2\lambda_p & 0 \\ 0 & \lambda_1\lambda_p \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} \tag{6}$$

Therefore, $\alpha'(s)/\beta'(s) = (\lambda_2/\lambda_1)\alpha(s)/\beta(s)$.

If the constant scale factor $\lambda_2/\lambda_1$ were known, this would give an invertible mapping — a correspondence — between lines in the two pencil spaces. This is true because $\alpha'(s)/\beta'(s)$ and $\alpha(s)/\beta(s)$ each uniquely determine a line in $P(\mathbf{l}'_1, \mathbf{l}'_2)$ and $P(\mathbf{l}_1, \mathbf{l}_2)$ and $\lambda_2/\lambda_1$ determines the mapping between lines in the two pencils (Figure 2). This correspondence may then be used to establish correspondence between points on $C$ and $C'$. For any pencil space line $\mathbf{l}$ intersecting $C$ in an unique point, its corresponding line $\mathbf{l}'$ would necessarily intersect $C'$ in an unique point, thereby establishing correspondence between one point on each curve. Corresponding pencil space lines intersecting $C$ and $C'$ in multiple points would have a limited matching ambiguity, but these could be resolved through ordering and continuity constraints. Therefore, knowing $\lambda_2/\lambda_1$ would essentially solve the correspondence problem on the original curve. This in turn would allow calculation of the projective transform $\mathtt{T}$.

### 3.5 The Projectivity After the Affine Transformations

Solving for the ratio $\lambda_2/\lambda_1$ can not be done using the pencil spaces alone. Determining this ratio requires using the second view of the mapping from curve points $\mathbf{p}$ to $[\alpha, \beta]^T$. In this view, as defined by (3), $\mathbf{q} = [\alpha, \beta, 1]^T$ is the affine transformation of $\mathbf{p}$ determined by the lines $\mathbf{l}_1$ and $\mathbf{l}_2$. Similarly, $\mathbf{q}' = [\alpha', \beta', 1]^T$ is the affine transformation of $\mathbf{p}'$ determined by the lines $\mathbf{l}_1'$ and $\mathbf{l}_2'$. The effect of these transformations of the projectivity $\mathtt{T}$ allows the ratio $\lambda_2/\lambda_1$, the point correspondences, and $\mathtt{T}$ itself to be found simultaneously and efficiently.

The affine transformations, represented by matrices $\mathtt{L}$ and $\mathtt{L}'$, along with equation (4), yield the following.

$$\begin{bmatrix} \alpha' \\ \beta' \\ 1 \end{bmatrix} = \mathtt{L}'\mathbf{p}' = \mathtt{L}'\lambda_p \mathtt{T}\mathbf{p} = \mathtt{L}'\lambda_p \mathtt{T}\mathtt{L}^{-1} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix}$$

Algebraic manipulation of this using

$$\mathtt{L} = \begin{bmatrix} -\mathbf{l}_2^T \\ \mathbf{l}_1^T \\ 0 \;\; 0 \;\; 1 \end{bmatrix} \quad \text{and} \quad \mathtt{L}' = \begin{bmatrix} -\mathbf{l}_2'^T \\ \mathbf{l}_1'^T \\ 0 \;\; 0 \;\; 1 \end{bmatrix}$$

and (5) shows that

$$\begin{bmatrix} \alpha' \\ \beta' \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} = \mathtt{T}_{\alpha\beta} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} \tag{7}$$

where equality here is defined up to an arbitrary scale factor. The third row of $\mathtt{T}_{\alpha\beta}$ is $[s_1, s_2, s_3]^T = \mathbf{t}_3^T \mathtt{L}^{-1}$ where $\mathbf{t}_3$ is the third row of $\mathtt{T}$. $\mathtt{T}_{\alpha\beta}$ has four degrees of freedom since it is determined up to a scale factor. Comparing (7) to (6) shows that the unknown scale factor in (6) is $\lambda_p = 1/(s_1\alpha + s_2\beta + s_3) = \mathbf{t}_3^T\mathbf{p}$.

$\mathtt{T}_{\alpha\beta}$ has several important properties. First, the parameters determining the ratio $\lambda_2/\lambda_1$, and indirectly the correspondence between points on the curves, are made explicit. Second, correspondence can also be determined by finding the nearest point $\mathbf{q}'$ to $\mathtt{T}_{\alpha\beta}\mathbf{q}$. Third, the original projectivity $T$ is immediately available as $\mathtt{L}'^{-1}\mathtt{T}_{\alpha\beta}\mathtt{L}$. Finally, when $T$ is affine, $[s_1, s_2, s_3] = [0, 0, 1]$. In this case, the transformation of $C'(s)$, described by points $\mathbf{q}'(s)$, is an anisotropic scaling of the transformation of curve $C(s)$, described by the points $\mathbf{q}(s)$. This makes calculation of $\lambda_1$ and $\lambda_2$ straightforward.

### 3.6 Computing Correspondence and $\mathtt{T}_{\alpha\beta}$

The properties of $\mathtt{T}_{\alpha\beta}$ and of the pencil space relationship $\alpha'/\beta' = (\lambda_2/\lambda_1)\alpha/\beta$ lead to a natural method of computing correspondence between curves and computing the parameters of $\mathtt{T}_{\alpha\beta}$. This is done using a process similar in nature to iterative closest point algorithms [1, 6]. The following description assumes discrete sets of transformed points $\mathbf{q}_i = [\alpha_i, \beta_i, 1]^T$ and $\mathbf{q}_j' = [\alpha_j', \beta_j', 1]^T$.

1. Compute an initial estimate of $\lambda_1$ and $\lambda_2$ using the affine approximation. This is currently done using a Hausdorff metric and search [8] for the two scale parameters. Initialize $s_1 = s_2 = 0$ and $s_3 = 1$.

2. Determine the new point correspondences. This can be done in two ways. First, the current estimate of the ratio $\lambda_2/\lambda_1$ can be used to establish correspondence between a pair of pencil lines, which in turn establishes correspondence between the curve points that the pencil lines intersect (Section 3.4). Second, for each $i$, the closest point $\mathbf{q}_j'$ to $\mathtt{T}_{\alpha\beta}\mathbf{q}_i$, for the current estimate of $\mathtt{T}_{\alpha\beta}$, determines a matched pair. The second method turns out to be more stable and is used in the current implementation.

3. Re-estimate the parameters of $\mathtt{T}_{\alpha\beta}$ based on the correspondences $\mathbf{q}_i, \mathbf{q}_j'$. Estimation uses a robust weighting scheme to reduce the effects of missing curve sections.

4. Repeat the previous two steps until the parameters of $\mathtt{T}_{\alpha\beta}$ converge.

### 3.7 Discussion of the Correspondence Technique

In order to calculate the projective transform, $\mathtt{T}$, between two planar curves, correspondence must be established between points on the curves. Unfortunately, a blind search over the eight degrees of freedom of $\mathtt{T}$ is untenable. The method proposed here starts from hypothesized correspondence between two pairs of lines that are tangent at inflection points of the curves. This correspondence reduces the degrees of freedom in $\mathtt{T}$ to four and leads to a natural, efficient iterative search method that simultaneously establishes correspondence between the curves and calculates the remaining parameters of $\mathtt{T}$. Intuitively, this can be thought of as a search for the single scale parameter $\lambda_2/\lambda_1$ embedded in the re-estimation of $\mathtt{T}_{\alpha\beta}$ (and therefore of $\mathtt{T}$) based on this parameter.

One item of potential concern is the stability of the tangent lines at inflection points. While the position of an inflection point is unstable, the tangent line, in contrast, is stable. To show this, let the curve $C(s) = [x(s), y(s)]^T$ be parameterized by arc length and have an inflection point at $s = 0$, and let $C(s)$ be described in a local coordinate frame with the origin at the inflection point and the $x$ axis tangent to $C$ at the origin (Figure 3). In this case, it is easy to show that up to the third order

$$x(s) = s + x_2 s^2 + x_3 s^3 \quad \text{and} \quad y(s) = y_3 s^3$$

The tangent line at the inflection is naturally stable because the curvature, $\kappa(s)$, is 0 at $s = 0$. This inherent stability increases as the second and third order coefficients of $x(s)$ and $y(s)$ decrease in magnitude. On the other hand, the stability of the inflection point location along the curve increases with $|\dot{\kappa}(0)| = |6y_3|$. Together, these observations indicate that instability of the inflection point leads to greater stability in the tangent line (Figure 3). When $|\dot{\kappa}(0)|$ is small, the curve is close to a line around the inflection point and the position of the inflection point is uncertain. This is exactly the case when the tangent line is most stable! Thus, although inflection point locations are unstable, tangent lines at inflection points are stable. The overall consequence is that estimating

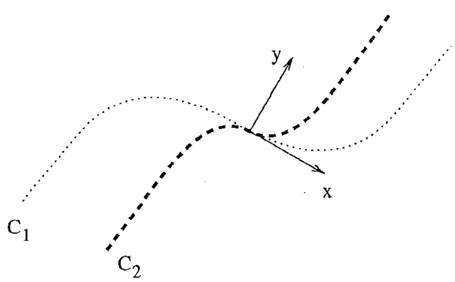


Figure 3: $C_1$ has a lower value of $y_3$ than $C_2$ and is approximately linear for a large interval around the inflection point. This make the inflection point location unstable but increases the stability of the tangent line.

$\mathtt{T}_{\alpha\beta}$ and converting back to T using L and L′ calculated from the line parameters should make T itself stable. To enhance stability, the estimated T can be used to rematch the original points on the curves, and from these correspondences T can be re-estimated.

## 4 Saliency of Symmetry Matches

In order to select appropriate curves for matching and to decide which curve pairs display a significant measure of symmetry it is necessary to define a saliency metric. For example, Cham and Cipolla use the smallest eigenvalue of the Hessian of the symmetry match error as a measure of the saliency of the match[5]. Their idea is that the localization of the symmetry match is a measure of saliency. That is, a significant match occurs when the match alignment can not be perturbed without large cost in curve fitting error.

Intuitively, a symmetry match will be salient if the curves are complex and if the transform between the curves is both accurate and unique. Consider a curve that is well approximated by a conic. This curve is simple, having no inflection points. Furthermore, a conic projectively maps onto any other conic[10], and there are three degrees of freedom to this projectivity. Therefore, symmetry matches between curves that are nearly conics should produce an extremely poor saliency measure. These observations lead to the following definition of saliency.

> The symmetry saliency, $S(C, C')$, of two curves, $C$ and $C'$ is defined as
>
> $$S(C, C') = \frac{\mathrm{Min}(\mathcal{E}(C, Q), \mathcal{E}(C', Q'))}{(\mathcal{E}(C', \mathtt{T}C) + \mathcal{E}(C, \mathtt{T}^{-1}C'))}, \qquad (8)$$
>
> where $\mathcal{E}(A, B)$ is the residual error of a least-squares fit of a curve $A$ to another curve $B$. Q is the best fitting conic to a curve segment. T is the projective transform that produces the best fit between two curves.

The numerator of (8) rates both the complexity of the matched curves and, indirectly, the uniqueness of the projective mapping. The denominator of (8) rates the accuracy of the mapping. Short curve segments are close to a conic and therefore eliminated from consideration. The remaining curve segments containing two or more inflection points are matched and the saliency measure of Equation 8 is computed for all pairs of corresponding inflection points. The curve-to-curve match and the projective transform T is computed according to the algorithm described in Section 3.6. The curve-to-conic match is carried out using Bookstein's algorithm[3]. The resulting matches are ranked according to the value of $S(C, C')$.

One aspect of the current approach that is somewhat ad hoc is the selection of the domain on each curve segment to consider as active for matching. In the current implementation, the curve spanning each pair of inflection points is extended, if possible, to include a total arc length of twice the length between the inflection points. The ideal approach will be to define a measure of curve complexity so that segments need only be extended to achieve enough structure to be uniquely matched. This refinement will be explored

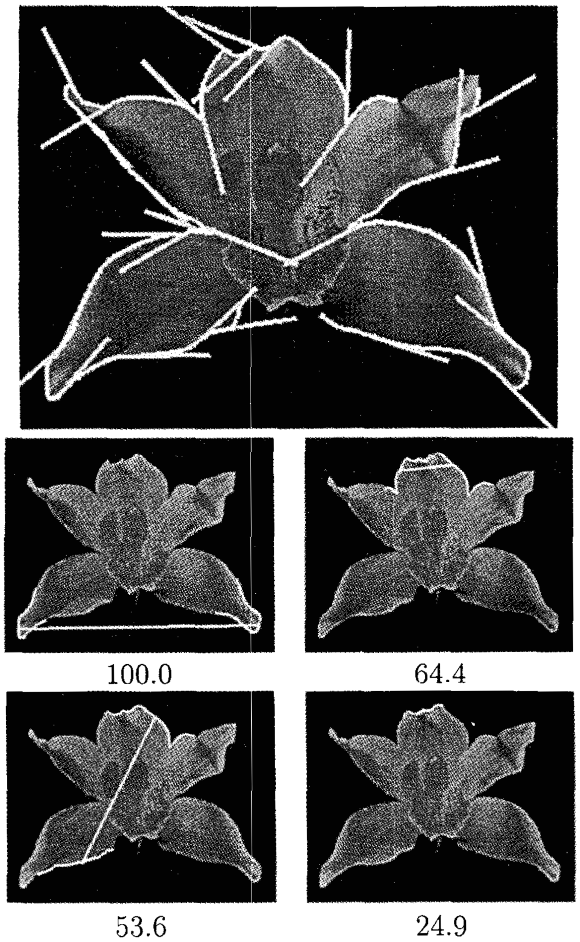


Figure 4: A series of matches in a scene containing an orchid. The outline of the orchid illustrate a good example of non-coplanar curve symmetry. The large top image shows the tangent lines at points of inflection used to form the correspondence mapping. The four smaller images show the top four saliency-ranked matching curve symmetries and the associated saliency value. The values are normalized so that the top ranked match has $S(C, C') = 100$. The line between curve segments indicates the matched curve pair, but is not a point correspondence.

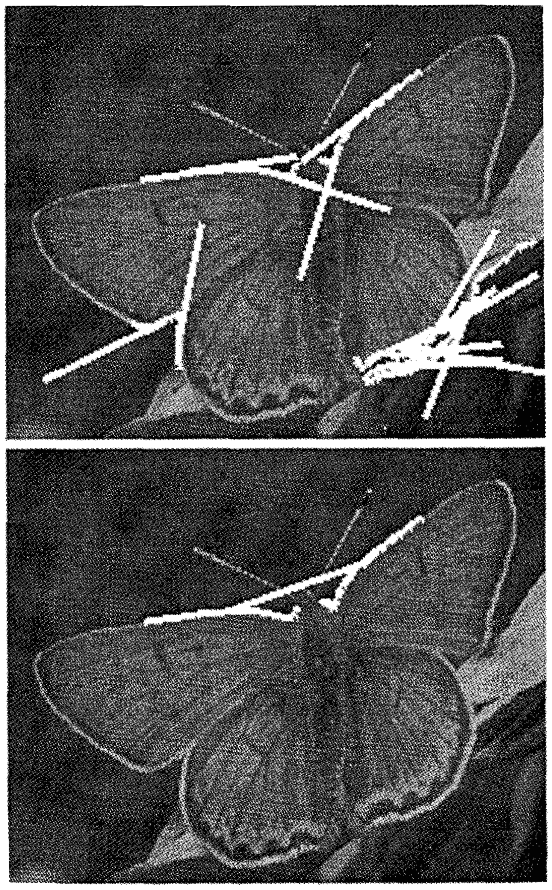

Figure 5: The top match for an image of a scene containing a butterfly. The top image shows the tangent lines at points of inflection used to form the correspondence mapping. The bottom image shows the top match which produced an order of magnitude higher saliency measure than any other match.

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## 5 Results

The algorithm has been applied to a number of scenes with good results. The saliency measure of Equation 8 produces curve matches with interesting symmetries in the top ranking of $S(C, C')$ values. Figure 4 illustrates the saliencies of the top four matching segments for an image of an orchid. Figure 5 illustrates the top curve match for an image of an butterfly.

## 6 Remarks

The effectiveness of the proposed saliency measure to capture plausible object symmetries has been demonstrated on reasonably complex and interesting scenes. A number of improvements can readily be made which should significantly improve both the selectivity of Equation 8 and the efficiency of the search for corresponding curves.

- The projective transform $\mathtt{T}$ should be constrained to prohibit the projection of the line at infinity onto or between curve pairs, thereby excluding matches from complex curves to nearly straight lines corresponding to the curve being projected to the horizon line.
- The current algorithm uses tangent lines at inflection points to establish initial correspondence, but still involves a combinatorial search of pairs of curves. A better approach would include the use of invariant local curve measures to prioritize and prune the search.
- The curves currently tested for saliency are unlikely to form complete boundaries of symmetric objects. Therefore, the ability to extend matching curve sections to cover the full symmetric intervals of the curves will be important to realizing the full potential of the techniques proposed here.

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